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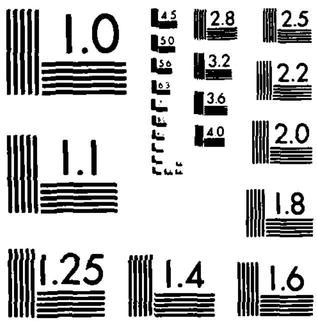

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ON INTERACTING POPULATIONS THAT DISPERSE TO AVOID CROWDING: PRESERVATION OF SEGREGATION

M. Bertsch, M. E. Gurtin,
D. Hilhorst, and L. A. Peletier

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

This paper discusses the dispersal of two interacting biological species. The dispersal - a response to population pressure alone - is modelled by the degenerate parabolic system

$$u_t = [u(u+v)_x]_x$$

$$v_t = k[v(u+v)_x]_x$$

in conjunction with an initial prescription of the individual densities u and v together with standard zero-flux boundary conditions. We demonstrate here the following interesting feature of this model: segregated initial data give rise to solutions which are segregated for all time.

AMS (MOS) Subject Classifications: 65P05, 92A15

Key Words: Degenerate parabolic equations, free-boundary problems, dispersal of biological populations.

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* Department of Mathematics, University of Leiden, Leiden, The Netherlands.

** Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213, USA.

*** CNRS, Laboratoire d'Analyse Numérique, Université de Paris-Sud, 91405 Orsay, France.

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SIGNIFICANCE AND EXPLANATION

This document ~~We~~ considers ~~here~~ a mathematical model for interacting biological species that disperse as a response to population pressure. The authors ~~We~~ demonstrate here an interesting feature of the model: species which are initially segregated remain segregated for all time.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON INTERACTING POPULATIONS THAT DISPERSE TO AVOID CROWDING: PRESERVATION OF SEGREGATION

M. Bertsch*, M. E. Gurtin**, D. Hilhorst***, and L. A. Peletier*

1. Introduction

Consider two interacting biological species with populations sufficiently dense that a continuum theory is applicable, and assume that the species are undergoing dispersal on a time scale sufficiently small that births and deaths are negligible. Granted these assumptions, conservation of population requires that

$$\begin{aligned} u_t &= -\operatorname{div}(uq), \\ v_t &= -\operatorname{div}(vw), \end{aligned} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are the spatial densities of the species, while the vector fields $q(x,t)$ and $w(x,t)$ are the corresponding dispersal velocities.

We restrict our attention to situations in which dispersal is a response to population pressure and express this mathematically by requiring that the dispersal of each of the species be driven by the gradient $\nabla(u+v)$ of the total population,[1] $u+v$. We therefore assume that

*Department of Mathematics, University of Leiden, Leiden, The Netherlands.

**Department of Mathematics, Carnegie-Mellon University, Pittsburgh, PA 15213, USA.

***CNRS, Laboratoire d'Analyse Numérique, Université de Paris-Sud, 91405 Orsay, France.

[1] For a single species this type of constitutive assumption was introduced by Gurney and Nisbet [1], Gurtin and MacCamy [2].

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$$q = -k_1 \nabla(u+v),$$
$$w = -k_2 \nabla(u+v), \tag{1.2}$$

with k_1, k_2 strictly-positive[1] constants, and this leads to the system[2]

$$u_t = k_1 \mathrm{div}[u\nabla(u+v)],$$
$$v_t = k_2 \mathrm{div}[v\nabla(u+v)]. \tag{1.3}$$

For convenience, we limit our attention to one space-dimension and we choose the time-scale so that $k_1 = 1$. Then writing $k = k_2$ we have the system

$$u_t = [u(u+v)_x]_x,$$
$$v_t = k[v(u+v)_x]_x. \tag{1.4}$$

We shall suppose that the two species live in a finite habitat

$$\Omega = (-L, L), \quad L > 0;$$

that individuals are unable to cross the boundary of Ω,

$$u(u+v)_x = v(u+v)_x = 0 \text{ for } x = \pm L, \quad t > 0; \tag{1.5}$$

and that the two populations are prescribed initially,

---

[1] The system (1.3) with $k_2 = 0$ was studied by Bertsch and Hilhorst [3] and by Bertsch, Gurtin, Hilhorst, and Peletier [4].

[2] Gurtin and Pipkin [5]. See also Busenberg and Travis [6]. An alternative theory was developed by Shigesada, Kawasaki, and Teramoto [7]. This theory is discussed in [4] and [5].

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{for} \quad x \in \Omega. \tag{1.6}$$

In this paper we shall study the problem (1.4)-(1.6) for <u>initial data which are segregated</u> in the sense that, for some $a \in \Omega$,

$$u_0(x) \equiv 0 \quad \text{for} \quad x > a, \quad v_0(x) \equiv 0 \quad \text{for} \quad x < a. \tag{1.7}$$

As our main result we establish the existence of a solution in which <u>the two species are segregated for all time</u>. This result is quite surprising[1] as it is independent of the initial distributions[2] of the species and of the ratio k of their dispersivities.

---

[1]Actually, Gurtin and Pipkin [5] gave a particular solution to (1.2) - *corresponding to* initial Dirac distributions - in which the two species are segregated for all time. Being a specific solution, it is not clear from this result whether "preservation of segregation" is a generic property of the equations (1.4).

[2]Granted they are segregated.

2. The problem. Results.

We shall use the notation

$$\mathbb{R}^+ = (0,\infty), \quad Q = \Omega \times \mathbb{R}^+, \quad Q_T = \Omega \times (0,T),$$

and, for any function $f: Q \to \mathbb{R}$,

$$Q^+(f) = \text{interior}\{(x,t) \in Q: \ f(x,t) > 0\}.$$

Our problem consists in finding functions $u(x,t)$ and $v(x,t)$ on $\bar{Q}$ such that

$$(\mathrm{I}) \begin{cases} \left.\begin{array}{l} u_t = [u(u+v)_x]_x \\ v_t = k[v(u+v)_x]_x \end{array}\right\} \text{ in } Q, & (2.1) \\ u(u+v)_x = v(u+v)_x = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, & (2.2) \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) \quad \text{in } \Omega. & (2.3) \end{cases}$$

We shall assume throughout that:

(A1) $k > 0$, $u_0, v_0 \geq 0$, $u_0, v_0 \in C(\bar{\Omega})$;

(A2) the initial data are segregated, so that (1.7) holds for some $a \in \Omega$;

(A3)[1] each of the sets $\{x: \ u_0(x) > 0\}$ and $\{x: \ v_0(x) > 0\}$ is connected.

The purpose of this paper is to establish - for such segregated initial data - solutions of (I) which are segregated for all time.

[1] We make this assumption for convenience only.

Proceeding formally, let (u,v) be a segregated solution. Then the sets $Q^+(u)$ and $Q^+(v)$ are disjoint; hence (assuming $u,v \geq 0$)

$$u \equiv 0 \quad \text{in} \quad Q^+(v), \qquad v \equiv 0 \quad \text{in} \quad Q^+(u),$$

and, by (2.1),

$$u_t = (uu_x)_x \quad \text{in} \quad Q^+(u), \quad v_t = k(vv_x)_x \quad \text{in} \quad Q^+(v). \tag{2.4}$$

Thus where positive u and v obey porous-media equations. As is well known,[1] solutions of the porous-media equation may not be smooth, and for that reason it is advantageous to work with a weak formulation of Problem (I). This is reinforced by the observation that (I) is a free-boundary problem and conditions at the free boundary are generally *indigenous to a weak formulation*, not required as separate restrictions. (The free boundary is the set

$$\mathfrak{F} = \{\partial Q^+(u) \cup \partial Q^+(v)\} \cap Q \tag{2.5}$$

which separates the region with $u(x,t) > 0$ from that with $u(x,t) = 0$ and separates the region with $v(x,t) > 0$ from that with $v(x,t) = 0$.)

With this in mind, assume for the moment that (u,v) is a smooth solution of (I). If we multiply (2.1) by an <u>arbitrary</u> smooth function $\psi(x,t)$, integrate over Q_T, and use (2.2)

---

[1] Cf., e.g., the survey article of Peletier [8].

and (2.3), we arrive at the relations

$$\int_{\Omega} \{u(T)\psi(T) - u_0\psi(0)\} = \int_{Q_T} \{u\psi_t - u(u+v)_x\psi_x\},$$
$$\int_{\Omega} \{v(T)\psi(T) - v_0\psi(0)\} = \int_{Q_T} \{v\psi_t - kv(u+v)_x\psi_x\}, \tag{2.6}$$

where we have used the notation $u(t) = u(\cdot,t)$, etc. We shall use (2.6) as the basis of our definition of a weak solution.

Definition. A (weak) solution of Problem (I) is a pair (u,v) with the following properties:

(i) $u,v \in L^{\infty}(Q_T)$ for $T > 0$; $u(t),v(t) \in L^{\infty}(\Omega)$ for $t \geq 0$; $(u+v)^2 \in L^2(0,T; H^1(\Omega))$ for $T > 0$;

(ii) $u(t),v(t) \geq 0$ almost everywhere in Ω for $t > 0$;

(iii) u and v satisfy (2.6) for all $\psi \in C^1(\bar{Q})$ and $T > 0$.

If, in addition, there is a continuous function ξ: $[0,\infty) \to \Omega$ such that, given any $t > 0$,

$$v(x,t) = 0 \quad \text{for} \quad -L < x < \xi(t),$$
$$u(x,t) = 0 \quad \text{for} \quad \xi(t) < x < L, \tag{2.7}$$

then (u,v) is segregated. We will refer to ξ as a separation curve.

Remarks:

1. The terms $u(u+v)_x$ and $v(u+v)_x$ in (2.6) are defined as follows:

$$u(u+v)_x = \begin{cases} \frac{1}{2}\frac{u}{u+v}\,[(u+v)^2]_x & \text{if } u > 0 \\ 0 & \text{if } u = 0 \end{cases}$$

and similarly for $v(u+v)_x$. Then, since $u/(u+v) \leq 1$, while $[(u+v)_-^2]_x \in L^2(Q_T)$, we have

$$u(u+v)_x \in L^2(Q_T) \quad \text{for} \quad T > 0.$$

2. The integral identities (2.6) imply that, as $t \to 0$,

$$u(t) \to u_0, \quad v(t) \to v_0 \quad \text{weakly in} \quad L^2(\Omega);$$

i.e., that

$$\int_\Omega [u(t)-u_0]\psi \to 0, \quad \int_\Omega [v(t)-v_0]\psi \to 0 \tag{2.8}$$

for all $\psi \in L^2(\Omega)$. To verify (2.8) we simply apply (2.6) with $\psi \in C^1(\bar{\Omega})$ (independent of time). This yields (2.8) for $\psi \in C^1(\bar{\Omega})$ and hence - using a standard argument - for $\psi \in L^2(\Omega)$.

3. The choice $\psi \equiv 1$ in (2.6) leads to the global conservation laws

$$\int_\Omega u(t) = \int_\Omega u_0, \quad \int_\Omega v(t) = \int_\Omega v_0 \tag{2.9}$$

for $t > 0$.

We close this section by stating our main results; the corresponding proofs will be given in Section 3.

Theorem 1. *Problem (I) has exactly one segregated solution.*

Remark. It is important to emphasize that we have established uniqueness only within the class of segregated solutions. Thus we have not ruled out the possibility - for segregated initial data - of solutions which mix. *We conjecture that this cannot happen.*

Theorem 2. *Let* (u,v) *be the segregated solution of Problem* (I). *Then:*

(i) $u + v \in C(\overline{Q})$;

(ii) $u_t = (uu_x)_x$ *classically*[1] *in* $Q^+(u)$;

(iii) $v_t = k(vv_x)_x$ *classically in* $Q^+(v)$.

---

[1]That is, u is C^∞ on $Q^+(u)$ and there satisfies $u_t = (uu_x)_x$ pointwise.

Our next theorem is concerned with the free boundary $\mathfrak{F}$ (cf. (2.5)). In view of (2.4), the portion of $\mathfrak{F}$ along which the two species are <u>not</u> in contact should have properties similar to those of the free boundary for the porous-media problem:

$$\text{(PM)}\begin{cases}\rho_t = (\rho\rho_x)_x & \text{in } Q,\\ \rho\rho_x = 0 & \text{on } \partial\Omega \times \mathbb{R}^+,\\ \rho(x,0) = \rho_0(x) & \text{in } \Omega.\end{cases}$$

As is known,[1] when the initial data have the form

$$\rho_0(x) > 0 \quad \text{in } (a_1,a_2), \qquad \rho_0(x) = 0 \quad \text{otherwise},$$

$-L < a_1 < a_2 < L$, the free boundary $Q \cap \partial Q^+(\rho)$ consists of two continuous, time-parametrizable curves, one emanating from a_1, one from a_2. If $b(t)$, $0 \le t < T_b$, designates the curve from a_1 (resp., a_2), then:

(F_1) $b(t) = b(0)$ on $(0,\tau_b)$ for some $\tau_b \in [0,T_b]$;

(F_2) $b(t)$ is C^1 and strictly decreasing (resp., strictly increasing) on (τ_b,T_b);

(F_3) $b(T_b^-) \in \partial\Omega$.

This discussion should motivate the following definition in which "FB" is shorthand for "free boundary".

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[1] Cf., e.g., the review article by Peletier [8]. See also Aronson and Peletier [9].

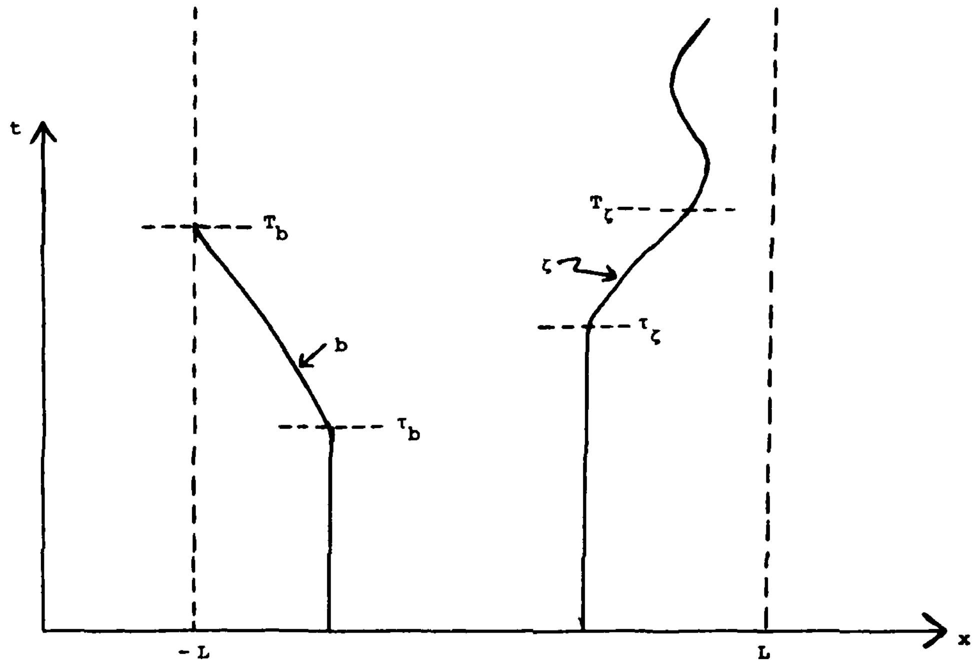

Figure 1. b is a left free-boundary curve extending to $\partial\Omega$. ζ is an internal free-boundary curve that is right up to time T_ζ.

Definition. An FB curve is a continuous function

$$b: [0,t_b) \to \Omega$$

(t_b may be ∞). Moreover:

(i) b is internal if $t_b = \infty$;

(ii) b is left (resp., right) up to time $T_b \in [0,t_b]$ if (F_1),(F_2) hold;

(iii) b extends to $\partial\Omega$ if b is right or left up to time t_b with $t_b < \infty$, and $b(t_b^-) \in \partial\Omega$.

Let $b: [0,t_b) \to \Omega$ be an FB curve and let $q: Q \to \mathbb{R}$. Then FB conditions with velocity q hold from the left (resp., right) on b if given any $t \in (0,t_b)$ at which b is C^1,

$$b'(t) = q(b(t)^+, t) \qquad (\text{resp., } b'(t) = q(b(t)^-, t)).$$

Theorem 3. Let (u,v) be the segregated solution of Problem (I). Then there exist FB curves $b_u, b_v, \zeta_u, \zeta_v$ with the following properties:

(i)[1] $b_u < \zeta_u \leq \zeta_v < b_v$ with b_u and ζ_u forming the boundary of $Q^+(u)$ in Q, b_v and ζ_v forming the boundary of $Q^+(v)$ in Q;

(ii) b_u and b_v extend to $\partial\Omega$, with b_u left and b_v right;

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[1] Here each inequality is assumed to hold at those times at which the underlying functions are defined.

(iii) ζ_u and ζ_v are internal, and there is a time $T \in [0,\infty)$ such that ζ_u is right up to time T, ζ_v is left up to time T, and

$$\zeta_u(t) = \zeta_v(t) =: \zeta(t) \quad \text{on } [T,\infty)$$

with $\zeta \in C^1(T,\infty)$; in addition,

$$(u+v)(\zeta(t),t) > 0 \quad \text{for} \quad t > T; \tag{2.10}$$

(iv) FB conditions with velocity $-u_x$ hold from the right on b_u, from the left on ζ_u;

(v) FB conditions with velocity $-kv_x$ hold from the right on ζ_v, from the left on b_v.

Remarks.

1. The curve ζ marks that portion of the free boundary on which the two species are in contact. By (2.10), the functions u and v suffer jump discontinuities across ζ; more precisely for $t > T$,

$$u(\zeta(t)^-,t) > 0, \quad u(\zeta(t)^+,t) = 0,$$

$$v(\zeta(t)^-,t) = 0, \quad v(\zeta(t)^+,t) > 0.$$

Further, (iii)-(v) of Theorem 3 in conjunction with the continuity of $u+v$ imply that, for $t > T$,

$$u(\zeta(t)^-, t) = v(\zeta(t)^+, t),$$

$$u_x(\zeta(t)^-, t) = kv_x(\zeta(t)^+, t) = -\zeta'(t). \tag{2.11}$$

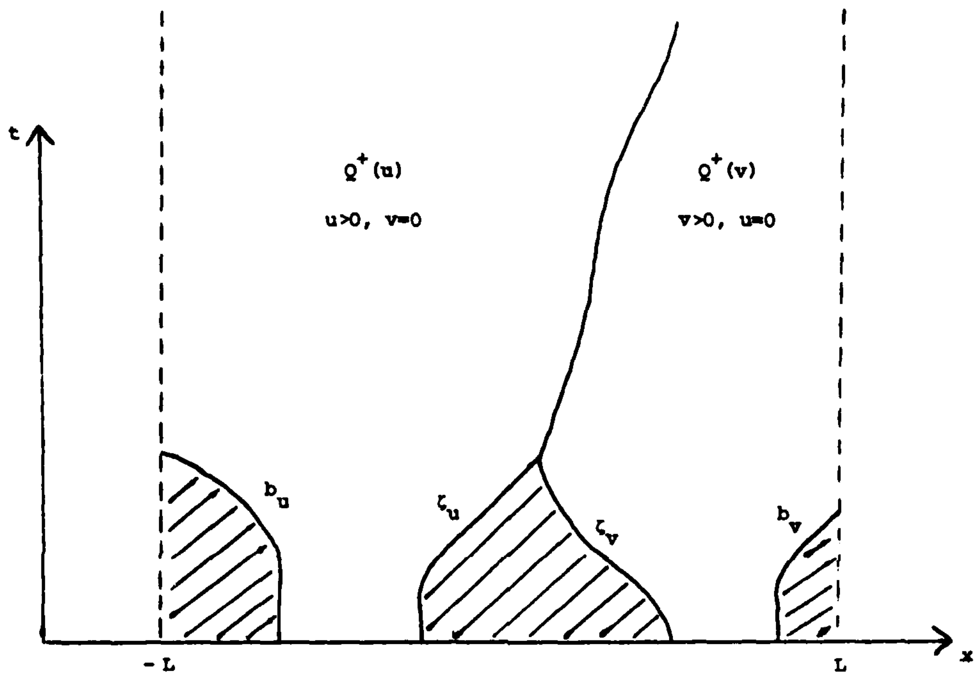

Figure 2. The free boundaries. The shaded areas correspond to $u = v = 0$.

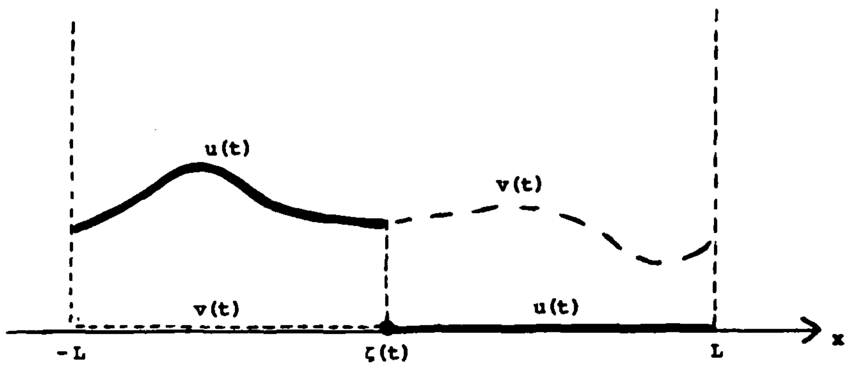

Figure 3. The functions $u(\cdot,t)$ and $v(\cdot,t)$ at a fixed time $t > T$.

2. The results (iv) and (v) assert that each of the "fronts" $b_u(t)$, $\zeta_u(t)$, $\zeta_v(t)$, $b_v(t)$ propagates with the velocity of individuals situated on it; condition (2.11) is the requirement that at the contact front $\zeta(t)$ the two species move together.

Our final result concerns the asymptotic behavior of segregated solutions. Proceeding formally, suppose that $u_\infty(x)$, $v_\infty(x)$ is an equilibrium solution of Problem (I). Then (2.1) and the boundary conditions (2.2) yield

$$u_\infty(u_\infty + v_\infty)' = v_\infty(u_\infty + v_\infty)' = 0 \quad \text{in} \quad \Omega;$$

hence

$$[(u_\infty + v_\infty)^2]' = 0 \quad \text{in} \quad \Omega$$

and

$$u_\infty + v_\infty = \text{constant.}$$

If u_∞ and v_∞ are segregated with habitats in $[-L, x_\infty]$ and $[x_\infty, L]$, respectively, then there exists a constant p such that

$$u_\infty(x) = \begin{cases} p & \text{if } x \in (-L, x_\infty) \\ \frac{1}{2}p & \text{if } x = x_\infty \\ 0 & \text{if } x \in (x_\infty, L) \end{cases} \tag{2.12}$$

$$v_\infty(x) = \begin{cases} 0 & \text{if } x \in (-L, x_\infty) \\ \frac{1}{2}p & \text{if } x = x_\infty \\ p & \text{if } x \in (x_\infty, L). \end{cases}$$

Moreover, if the equilibrium solution (u_∞, v_∞) is reached from the initial data (u_0, v_0), the conservation law (2.9) implies that

$$p = \frac{U + V}{2L}, \qquad x_\infty = \frac{U - pL}{p}, \tag{2.13}$$

where

$$U = \int_\Omega u_0, \qquad V = \int_\Omega v_0. \tag{2.14}$$

That these formal calculations are indeed correct is a consequence of the following theorem.

<u>Theorem</u> 4. <u>Let</u> (u,v) <u>be the segregated solution of</u> (I). <u>Then, as</u> $t \to \infty$,

$$\zeta(t) \to x_\infty, \quad u(t) \to u_\infty, \quad v(t) \to v_\infty,$$

<u>the latter two limits being pointwise in</u> $\Omega \backslash \{x_\infty\}$. <u>Here</u> $\zeta(t)$ <u>is the contact front</u> (cf. <u>Theorem</u> 3), <u>while</u> x_∞, u_∞, <u>and</u> v_∞ <u>are defined in</u> (2.12), (2.13).

3. Reformulation of the problem.

Let (u,v) be a sufficiently smooth segregated solution of Problem (I) and define

$$z(x,t) = -U + \int_{-L}^{x} [u(y,t) + v(y,t)]dy; \tag{3.1}$$

$z(x,t)$ represents the total population, at time t, in the interval $[-L,x]$. In view of the conservation laws (2.9),

$$z(-L,t) = -U, \quad z(\xi(t), t) = 0, \quad z(L,t) = V, \tag{3.2}$$

where U and V are defined in (2.14), so that separation curves $\xi(t)$ (cf. (2.7)) are level curves $z(\xi(t), t) = 0$; in fact,

$$\begin{aligned} z(x,t) &\leq 0 \quad \text{for} \quad x < \xi(t), \\ z(x,t) &\geq 0 \quad \text{for} \quad x > \xi(t). \end{aligned} \tag{3.3}$$

Further, if we differentiate (3.1) with respect to t and use (2.1), (2.2), and (2.7), we find that

$$z_t = \begin{cases} z_x z_{xx} & \text{for} \quad x < \xi(t) \\ k z_x z_{xx} & \text{for} \quad x > \xi(t). \end{cases} \tag{3.4}$$

Thus defining $c\colon \mathbb{R} \to \mathbb{R}$ by

$$c(s) = \begin{cases} s, & s \leq 0 \\ s/k, & s > 0, \end{cases} \tag{3.5}$$

we may use (3.3) to reduce (3.4) to the single equation

$$c(z)_t = z_x z_{xx}$$

on all of Q. Therefore, if we write

$$z_0(x) = -U + \int_{-L}^{x} (u_0+v_0), \tag{3.6}$$

we are led to the following problem for $z(x,t)$:

$$(\mathrm{II}) \begin{cases} c(z)_t = z_x z_{xx} \quad \text{in} \quad Q, \\ z(-L,t) = -U, \quad z(L,t) = V, \quad t > 0, \\ z(x,0) = z_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

We assume, for the remainder of the section, that c and z_0 are defined by (3.5) and (3.6), and that (A1)-(A3) are satisfied.

Problem (II) under hypotheses more general than ours, has been analyzed in [10]. We shall simply state, without proof, a version of the results of [10] appropriate for our use. With this in mind, we first define what we mean by a solution; in that definition, and in what follows, $z_\infty(x)$ designates the unique equilibrium solution of (II):

$$z_\infty(x) = \frac{(U+V)(x+L)}{2L} - U.$$

Definition. A (weak) solution of Problem (II) is a function $z \in C([0,\infty); W^{1,\infty}(\Omega))$ with the following properties:

(i) $z(\cdot,t) - z_\infty \in H_0^1(\Omega)$ for $t > 0$;

(ii) $z_t \in L^2(Q_T)$ for $T > 0$;

(iii) for all $\psi \in C^1(\bar{Q})$ with $\psi = 0$ on $\partial\Omega \times (0,\infty)$

and all $T > 0$,

$$\int_\Omega \{c(z(T))\psi(T) - c(z_0)\psi(0)\} = \int_{Q_T} \{c(z)\psi_t - \tfrac{1}{2}(z_x)^2\psi_x\}. \qquad (3.7)$$

Theorem 5 ([10]). Problem (II) has exactly one solution z. Moreover:

(i) $z_x \in C(\bar{Q})$ with $z_x \geq 0$;

(ii) $c(z)_t = z_x z_{xx}$ classically in $Q^+(z_x)$;

(iii) $Q^+(z_x)$ is the union of the sets

$$Q_1 := \{(x,t) \in Q\colon -U < z(x,t) < 0\},$$

$$Q_2 := \{(x,t) \in Q\colon 0 < z(x,t) < V\},$$

and there exist free-boundary curves $b_1, b_2, \zeta_1, \zeta_2$ such that (i) - (v) of Theorem 3 hold with $Q^+(u), Q^+(v), b_u, b_v, \zeta_u, \zeta_v$ replaced by $Q_1, Q_2, b_1, b_2, \zeta_1, \zeta_2$, respectively, with $u+v$ in (iii) replaced by z_x, and with u_x and v_x in (iv) and (v) replaced by z_{xx};

(iv) $z(t) \to z_\infty$ in $C^1(\bar{\Omega})$ as $t \to \infty$;

(v) given any $x_0 \in \Omega$ with $z_0'(x_0) > 0$, there exists a continuous function $\xi: [0,\infty) \to \Omega$ such that

$$\{(x,t) \in \bar{Q}: \ z(x,t) = z_0(x_0)\} = \{(x,t) \in \bar{Q}: \ x = \xi(t)\};$$

moreover, $\xi \in C^1(0,\infty)$ and, for $t > 0$,

$$\xi'(t) = -\kappa z_{xx}(\xi(t), t),$$

where $\kappa = 1$ or k according as $z_0(x_0) < 0$ or $z_0(x_0) > 0$.

The next result asserts the equivalence of Problems I and (II) and, when combined with Theorem 5, yields the validity of Theorems 1-4.

Theorem 6. Problems (I) and (II) are equivalent:

(i) Let z be a solution of Problem (II) and define u and v on $\bar{Q}$ by

$$\begin{aligned} &u(x,t) = z_x(x,t), \quad v(x,t) = 0 \quad \text{if} \quad z(x,t) < 0, \\ &u(x,t) = 0, \quad v(x,t) = z_x(x,t) \quad \text{if} \quad z(x,t) > 0, \\ &u(x,t) = v(x,t) = \tfrac{1}{2} z_x(x,t) \quad \text{if} \quad z(x,t) = 0. \end{aligned} \tag{3.8}$$

Then (u,v) is a segregated solution of Problem (I).

(ii) Conversely, let (u,v) be a segregated solution of Problem (I) and define z on $\bar{Q}$ by (3.1). Then z solves Problem (II).

Proof.

(i) Let z be a solution of (Π) and define (u,v) through (3.8). By Theorem 5(i), the only nontrivial step in showing that (u,v) solves (I) is proving that u and v satisfy the integral identities (2.6). We shall only verify the first of (2.6); the verification of the second is completely analogous.

For convenience, we write $b = b_1$, $\zeta = \zeta_1$ for the FB curves established in Theorem 5, and we extend $b(t)$ continuously to $[0,\infty)$ by defining $b(t) = -L$ for $t \geq t_b$. By Theorem 5(iii) and (3.8),

$$\operatorname{supp} u(t) = [b(t), \zeta(t)].$$

Choose $\varepsilon > 0$ sufficiently small and let $b_\varepsilon(t)$ and $\zeta_\varepsilon(t)$, respectively, be the level curves $z = -U + \varepsilon$ and $z = -\varepsilon$ (cf. Theorem 5(v)). Then, by Theorem 5(ii) and (3.8), $u_t = (uu_x)_x$ classically and $v \equiv 0$ must both be satisfied in a neighborhood of any (x,t) such that $b_\varepsilon(t) \leq x \leq \zeta_\varepsilon(t)$ and $t > 0$. Further, Theorem 5(v) yields

$$b_\varepsilon'(t) = -u_x(b_\varepsilon(t), t), \quad \zeta_\varepsilon'(t) = -u_x(\zeta_\varepsilon(t), t)$$

for $t > 0$. Thus, choosing $\delta > 0$, the identity

$$\int_\delta^t f'(\tau)d\tau = f(t) - f(\delta)$$

applied to

$$f(\tau) = \int_{b_\varepsilon(\tau)}^{\zeta_\varepsilon(\tau)} u(\tau)\psi(\tau)$$

yields, when $\psi \in C^1(\bar{Q})$,

$$\int_{b_\varepsilon(t)}^{\zeta_\varepsilon(t)} u(t)\psi(t) - \int_{b_\varepsilon(\delta)}^{\zeta_\varepsilon(\delta)} u(\delta)\psi(\delta) = \int_\delta^t \left\{ \int_{b_\varepsilon(\tau)}^{\zeta_\varepsilon(\tau)} (u\psi_\tau - u(u+v)_x\psi_x)\,dx \right\} d\tau. \quad (3.9)$$

Next, since $b_\varepsilon(t) \downarrow b(t)$ and $\zeta_\varepsilon(t) \uparrow \zeta(t)$ as $\varepsilon \downarrow 0$ for each $t \in [0,\infty)$, it follows from Lebesgue's dominated convergence theorem that (3.9) holds with b_ε and ζ_ε replaced by b and ζ. Also, since $z_x \in C(\bar{Q})$, it follows that $z_x(\delta) \to z_0'$ in $C(\bar{\Omega})$ as $\delta \downarrow 0$ and

$$\int_{b(\delta)}^{\zeta(\delta)} u(\delta)\psi(\delta) \to \int_{b(0)}^{\zeta(0)} u_0\psi(0) \quad \text{as} \quad \delta \downarrow 0.$$

Thus a second application of Lebesgue's theorem yields

$$\int_{b(t)}^{\zeta(t)} u(t)\psi(t) - \int_{b(0)}^{\zeta(0)} u_0\psi(0) = \int_0^t \left\{ \int_{b(\tau)}^{\zeta(\tau)} (u\psi_\tau - u(u+v)_x\psi_x)\,dx \right\} d\tau,$$

and, since $u(t) \equiv 0$ on $\Omega\backslash(b(t), \zeta(t))$, the first of (2.6) follows.

(ii) Let (u,v) solve (I). We are to prove that z - defined by (3.1) - solves (II). Choose $T > 0$. Then $u,v \in L^\infty(Q_T)$ and hence, by (2.9) and the definition of z_∞, $z(\cdot,t) - z_\infty \in H_0^1(\Omega)$. Note also that, since $z_x = u + v$ and $(u+v)^2 \in L^2(0,T;H^1(\Omega))$, it follows that

$$z_x^2 \in L^2(0,T;H^1(\Omega)), \tag{3.10}$$

where $z_x^2 = (z_x)^2$.

Our next step will be to establish the integral identity (3.7). Thus choose $\chi \in C^1(\bar{Q})$ with $\chi = 0$ on $\partial\Omega \times (0,\infty)$ and take

$$\psi(x,t) = \int_{-L}^{x} \chi(y,t)\,dy$$

in (2.6); in view of (2.7), (3.1), and (3.6), the result is

$$\begin{aligned} \int_{-L}^{\xi(T)} z_x(T)\psi(T) - \int_{-L}^{\xi(0)} z_0'\psi(0) &= \int_0^T \int_{-L}^{\xi(t)} \{z_x\psi_t - \tfrac{1}{2}(z_x^2)_x\chi\}dxdt, \\ \int_{\xi(T)}^{L} z_x(T)\psi(T) - \int_{\xi(0)}^{L} z_0'\psi(0) &= \int_0^T \int_{\xi(t)}^{L} \{z_x\psi_t - \tfrac{k}{2}(z_x^2)_x\chi\}dxdt. \end{aligned} \tag{3.11}$$

Since $\psi_x = \chi$, $\psi(-L,t) = 0$, and $\chi(\pm L,t) = 0$, while $z(x,t)$ satisfies (3.2) and (3.3), if we integrate (3.11) by parts and then add the first of the resulting relations to k^{-1} times the second, we arrive at (3.7) (with ψ replaced by χ).

We have only to show that $z_t \in L^2(Q_T)$. But this follows from (3.10) and the fact that, by (3.7), $c(z)_t = \frac{1}{2}(z_x^2)_x$ in the sense of distributions on Q_T. This completes the proof of Theorem 6.

4. Remarks. Open problems. Conjectures.

1. Problem (I) with nonsegregated initial data is open. Here the problem does not reduce to a free boundary problem for a single scalar field z, as one must solve the system (2.1) in regions of interaction (cf. Remark 2).

2. The system (2.1) with $k = 1$ is far simpler to analyze. There the total density $\rho = u + v$ satisfies (PM) with initial data $\rho_0 = u_0 + v_0$, and once ρ is known (2.1) are linear hyperbolic equations for u and v:

$$u_t = (u\rho_x)_x, \quad v_t = (v\rho_x)_x$$

(cf. [5]). Using this reduction one can prove uniqueness within the class of all solutions (as opposed to all segregated solutions), and one can show that solutions which begin mixed remain mixed for all time, including $t = \infty$. (Details will appear elsewhere.)

3. Assume, in place of (1.2), that the dispersal of each of the species is driven by a weighted sum of the densities; i.e., that (in one space-dimension),

$$\begin{aligned} q &= -(k_{11}u + k_{12}v)_x \\ w &= -(k_{21}u + k_{22}v)_x \end{aligned} \tag{4.1}$$

with all

$$k_{ij} > 0. \tag{4.2}$$

This constitutive assumption, when combined with the conservation law (1.1) and corresponding zero-flux boundary conditions, leads to the problem

$$(\mathrm{III})\begin{cases} u_t = [u(k_{11}u + k_{12}v)_x]_x \\ v_t = [v(k_{21}u + k_{22}v)_x]_x \end{cases} \text{ in } Q, \qquad (4.3)$$

$$\begin{cases} u(k_{11}u + k_{12}v)_x = v(k_{21}u + k_{22}v)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

This formulation is greatly simplified if we define new independent variables

$$\alpha(x,t) = k_{11}u(x,t), \quad \beta(x,t) = k_{12}v(x,t)$$

and new constants

$$k = \frac{k_{22}}{k_{12}}, \qquad \mu = \frac{k_{12}k_{21}}{k_{11}k_{22}},$$

for then (III) reduces to

$$(\mathrm{IV})\begin{cases} \alpha_t = [\alpha(\alpha+\beta)_x]_x \\ \beta_t = k[\beta(\mu\alpha + \beta)_x]_x \end{cases} \text{ in } Q, \qquad (4.4)$$

$$\begin{cases} \alpha(\alpha + \beta)_x = \beta(\mu\alpha + \beta)_x = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ \alpha(x,0) = \alpha_0(x), \quad \beta(x,0) = \beta_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

with

$$\alpha_0 = k_{11}u_0, \quad \beta_0 = k_{12}v_0.$$

4. Consider Problem (III), or equivalently (IV). The terms on the right side of (4.3) involving second derivatives are

$$\begin{pmatrix} uk_{11} & uk_{12} \\ vk_{21} & vk_{22} \end{pmatrix} \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}. \tag{4.5}$$

Writing $A = A(u,v)$ for the coefficient matrix in (4.5) and confining our attention to $u > 0$, $v > 0$, we conclude from (4.2) that there are exactly three possibilities for the eigenvalues $\lambda_1 \leq \lambda_2$ of A, namely:

(i) $\lambda_1 > 0$, $\lambda_2 > 0$; (ii) $\lambda_1 = 0$, $\lambda_2 > 0$; (iii) $\lambda_1 < 0$, $\lambda_2 > 0$.

Moreover, writing K for the matrix

$$K = (k_{ij}),$$

it is not difficult to verify that

(i) $\lambda_1 > 0, \quad \lambda_2 > 0 \iff \det K > 0,$

(ii) $\lambda_1 = 0, \quad \lambda_2 > 0 \iff \det K = 0,$

(iii) $\lambda_1 < 0, \quad \lambda_2 > 0 \iff \det K < 0.$

We consider the three cases separately.

Case (i) ($\det K > 0$). Here the system (4.3) is degenerate parabolic, as it is parabolic when $u > 0$ and $v > 0$, but not when $uv = 0$. Because of this property, we expect that initially-segregated solutions will eventually mix. We also expect them to mix for another reason. Indeed, assume to the contrary that Problem (III)

has a segregated solution (u,v). For such a solution we would expect the two populations to spread until they meet, and then to remain in contact along a contact front $\zeta(t)$ (cf. Theorem 3 and Remark 2 following it). From (4.2) one might expect that both $k_{11}u + k_{12}v$ and $k_{21}u + k_{22}v$ would be continuous across ζ, and hence both zero along ζ, a condition which cannot generally be satisfied (cf. Remark 1 following Theorem 3). We are therefore led to the following conjecture: <u>for</u> $\det K > 0$ <u>there are no segregated solutions of Problem</u> (III). In this regard it would be interesting to look at (III) with[1]

$$K = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{pmatrix},$$

$\varepsilon > 0$; in particular, the limit $\varepsilon \to 0$.

Finally, within the context of the biological model, the off-diagonal elements of K drive the segregation of the species, while the diagonal elements, by themselves, result in the usual diffusive behavior. Since $\det K > 0$ yields $k_{11}k_{22} > k_{12}k_{21}$, it would seem reasonable that in this case the two species ultimately mix.

<u>Case</u> (ii) ($\det K = 0$). Here $\mu = 1$ and Problem (IV) is identical to Problem (I). Thus <u>all of our results generalize trivially to populations whose interaction is described by</u> (4.1) <u>with</u> K <u>singular</u>.

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[1]This choice of K arose in discussions with R. Rostamian.

For the case $\det K = 0$ we would like to call the system (4.3) degenerate parabolic-hyperbolic. Indeed, if we set $w = \alpha + \beta$, then, assuming $k \geq 1$, (4.4) can be written as

$$w_t = [\{w + (k-1)\beta\}w_x]_x,$$

$$\beta_t = k(w_x\beta)_x$$

i.e., as a system composed of a degenerate-parabolic equation and a hyperbolic equation. The presence of this last equation makes the discontinuity of u and v at the contact front less surprising.

In this case one can speak of "passive segregation": if the species start segregated, they may remain segregated, as we have seen in the previous sections, and if they start mixed, then, when $k = 1$, they remain mixed for all $t \geq 0$ (see Remark 2 of this section).

Case (iii) ($\det K < 0$). The system (4.3) is now not parabolic, and Problem (II) is probably not well posed. Since the off-diagonal terms in K dominate in this case, one might expect a tendency towards segregation, even in a mixed population.

5. The system (2.1) with $k = 0$ was studied in [3] and [4]. There $v(x) \equiv v_0(x)$ and the problem reduces to solving $(2.1)_1$, $(2.2)_1$, and $(2.3)_1$. In this case, even with segregated initial data, solutions eventually mix, an apparent contradiction in behavior. The limit $k \to 0$ in Problem (I) would therefore be interesting.

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This paper discusses the dispersal of two interacting biological species. The dispersal - a response to population pressure alone - is modelled by the degenerate parabolic system

$$u_t = [u(u+v)_x]_x$$

$$v_t = k[v(u+v)_x]_x$$

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ABSTRACT (continued)

in conjunction with an initial prescription of the individual densities u and v together with standard zero-flux boundary conditions. We demonstrate here the following interesting feature of this model: <u>segregated initial data give rise to solutions which are segregated for all time</u>.

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